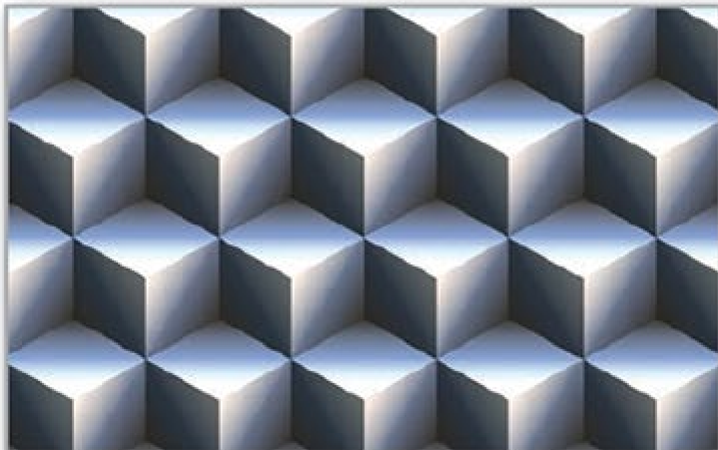
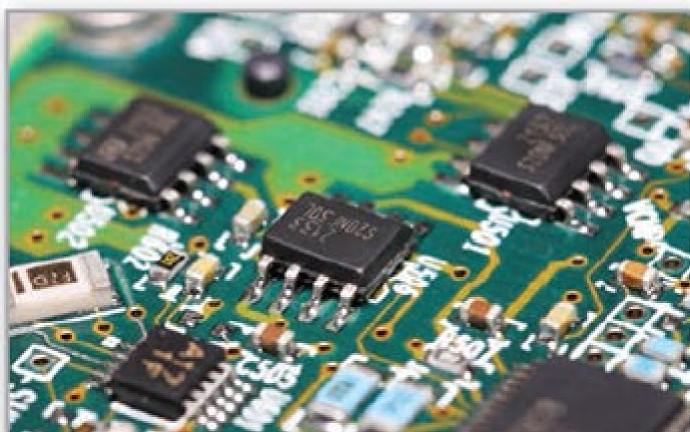
6 Linear Transformations





6.5

Applications of Linear Transformations

Objectives

- Identify linear transformations defined by reflections, expansions, contractions, or shears in R^2.

- Use a linear transformation to rotate a figure in R^3.

The Geometry of Linear Transformations in R^2

The Geometry of Linear Transformations in R^2 (1 of 2)

The first part of this section gives geometric interpretations of linear transformations represented by 2 × 2 elementary matrices.

Elementary Matrices for Linear Transformations in R^2

Reflection in y-Axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-Axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The Geometry of Linear Transformations in R^2 (2 of 2)

Elementary Matrices for Linear Transformations in R^2

Reflection in Line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Horizontal Expansion ($k > 1$)
or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical Expansion ($k > 1$)
or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Horizontal Shear

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

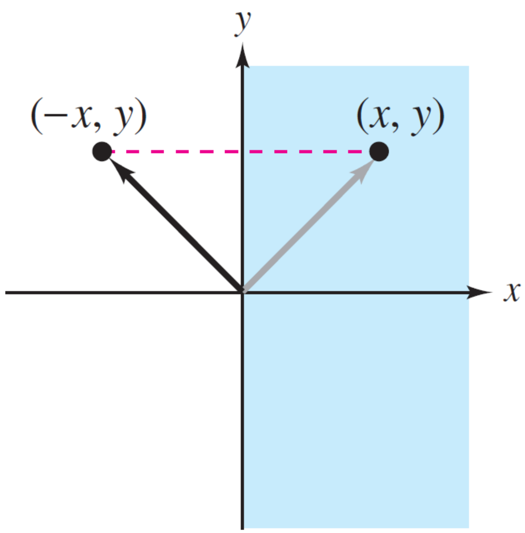
Example 1 – Reflections in R^2 (1 of 3)

The transformations below are **reflections.** These have the effect of mapping a point in the *xy*-plane to its "mirror image" with respect to one of the coordinate axes or the line $y = x$, as shown in Figure 6.5.

a. Reflection in the *y*-axis:

$$T(x, y) = (-x, y)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



Reflections in R^2

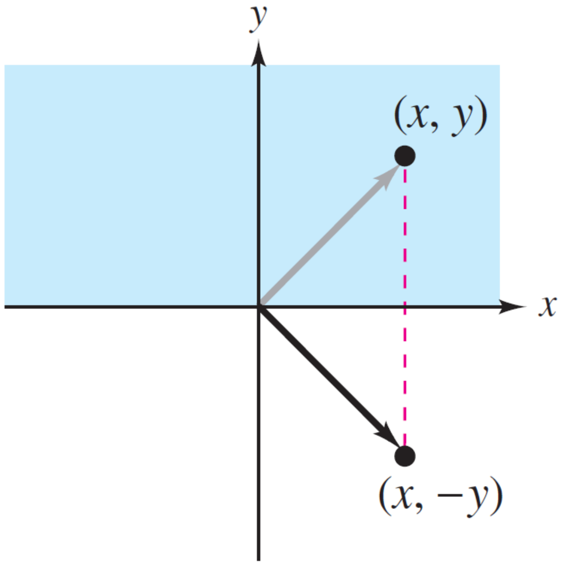
Figure 6.5(a)

Example 1 – Reflections in R^2 (2 of 3)

b. Reflection in the *x*-axis:

$$T(x, y) = (x, -y)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



Reflections in R^2

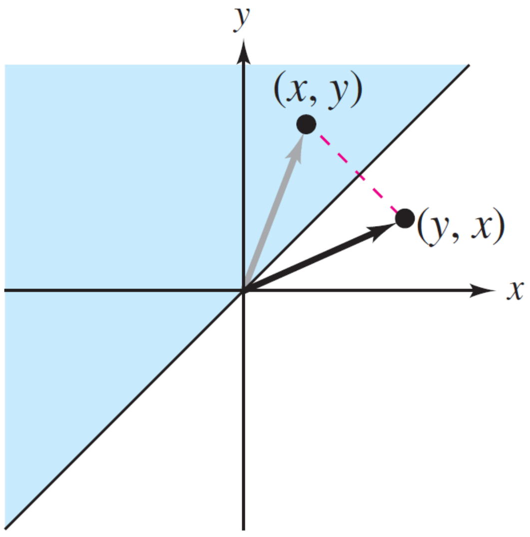
Figure 6.5(b)

Example 1 – Reflections in R^2 (3 of 3)

c. Reflection in the line $y = x$:

$$T(x, y) = (y, x)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



Reflections in R^2

Figure 6.5(c)

Example 2 – Expansions and Contractions in R^2 (1 of 3)

The transformations below are **expansions** or **contractions,** depending on the value of the positive scalar *k*.

a. Horizontal expansions and contractions:

$$T(x, y) = (kx, y)$$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$

b. Vertical expansions and contractions:

$$T(x, y) = (x, ky)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$$

Example 2 – Expansions and Contractions in R^2 (2 of 3)

Note in the figures below that the distance the point (x, y) moves by a contraction or an expansion is proportional to its x- or y-coordinate. For example, under the transformation represented by

$$T(x, y) = (2x, y)$$

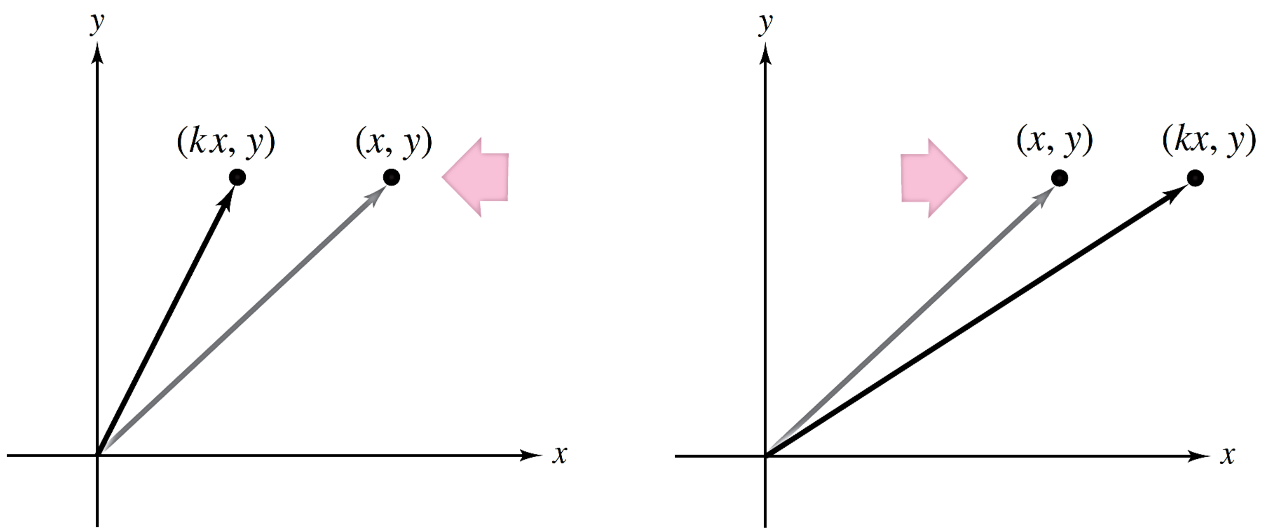
the point (1, 3) would move one unit to the right, but the point (4, 3) would move four units to the right.

Example 2 – Expansions and Contractions in R^2 (3 of 3)

Under the transformation represented by

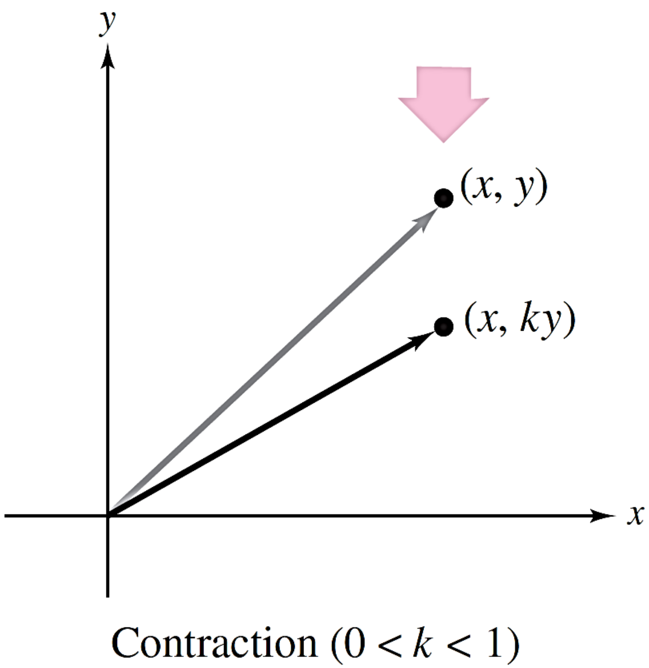
$$T(x, y) = \left(x, \tfrac{1}{2}y\right)$$

the point (1, 4) would move two units down, but the point (1, 2) would move one unit down.



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

Example 2 – Expansions and Contractions in R^2 (4 of 3)



Example 3 – Shears in R^2 (1 of 3)

The transformations below are **shears.**

$T(x, y) = (x + ky, y)$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

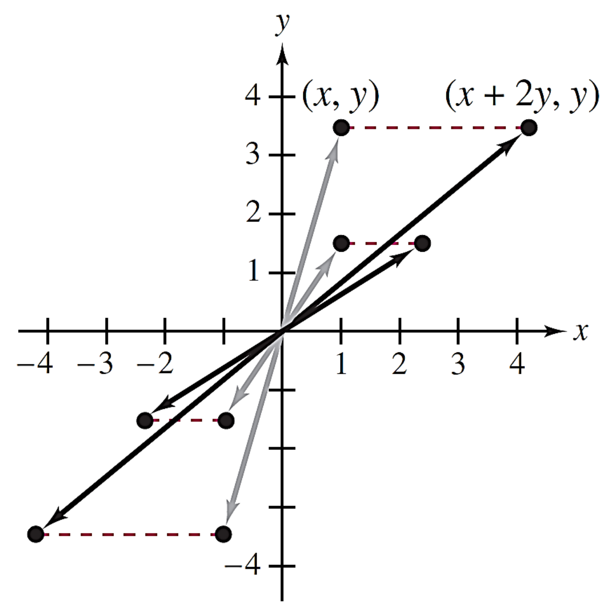
$T(x, y) = (x, y + kx)$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

a. A horizontal shear represented by

$T(x, y) = (x + 2y, y)$

is shown at the right. Under this transformation, points in the upper half-plane "shear" to the right by amounts proportional to their y-coordinates.



Example 3 – Shears in R^2 (2 of 3)

Points in the lower half-plane "shear" to the left by amounts proportional to the absolute values of their y-coordinates. Points on the x-axis do not move by this transformation.

b. A vertical shear represented by

$$T(x, y) = (x, y + 2x)$$

is shown below. Here, points in the right half-plane "shear" upward by amounts proportional to their x-coordinates.

Example 3 – Shears in R^2 (3 of 3)

Points in the left half-plane "shear" downward by amounts proportional to the absolute values of their *x*-coordinates. Points on the *y*-axis do not move.