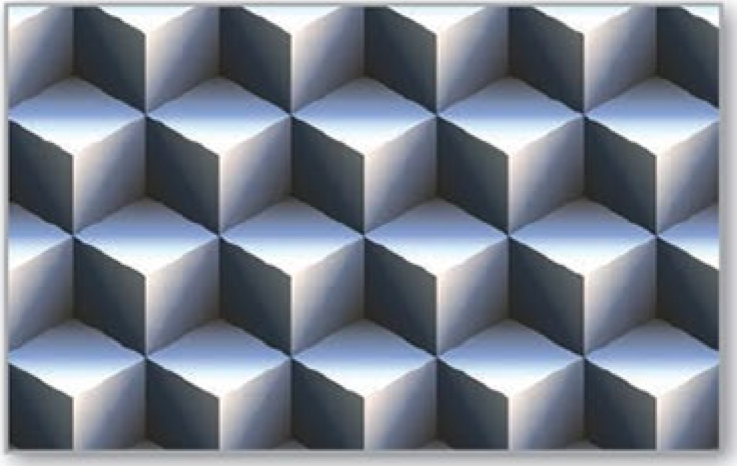
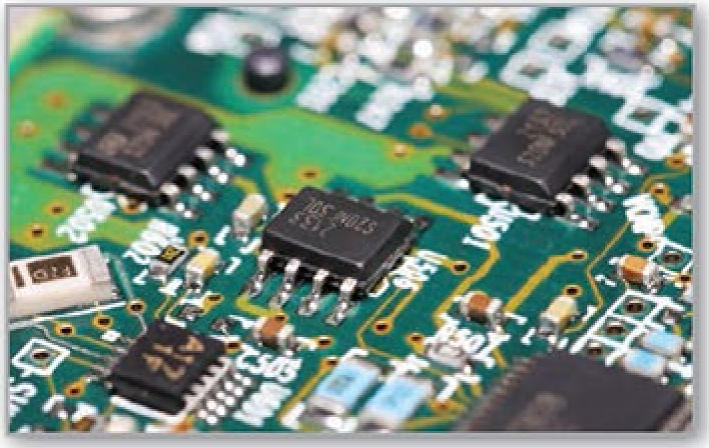
6 Linear Transformations




6.1 Introduction to Linear Transformations

Objectives

- Find the image and preimage of a function.
- Show that a function is a linear transformation, and find a linear transformation.

Images And Preimages of Functions

Images and Preimages of Functions (1 of 2)

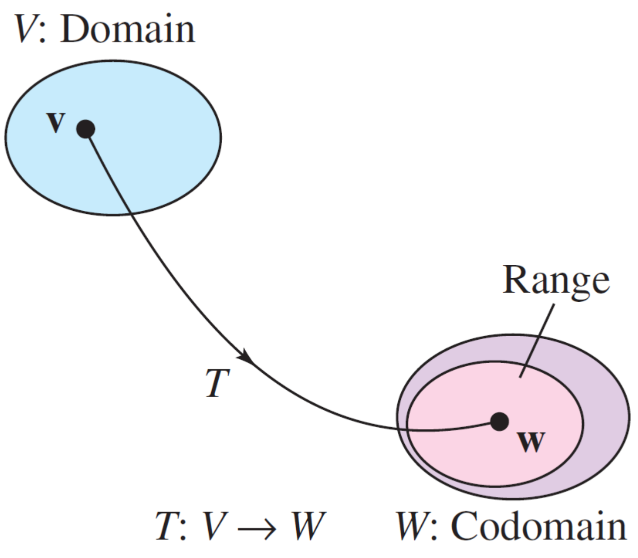
In this section, we will learn about functions that **map** a vector space *V* into a vector space *W*. This type of function is denoted by

$$T: V \to W.$$

The standard function terminology is used for such functions. For instance, *V* is the **domain** of *T*, and *W* is the **codomain** of *T*.

Images and Preimages of Functions (2 of 2)

If **v** is in V and **w** is in W such that $T(\mathbf{v}) = \mathbf{w}$, then **w** is the **image** of **v** under T. The set of all images of vectors in V is the **range** of T, and the set of all **v** in V such that $T(\mathbf{v}) = \mathbf{w}$ is the **preimage** of **w**. (See below.)



Example 1 – A function from R^2 into R^2

For any vector **v** = (v_1, v_2) in R^2, define $T: R^2 \rightarrow R^2$ by

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2).$$

a. Find the image of **v** = (−1, 2).

b. Find the image of **v** = (0, 0).

c. Find the preimage of **w** = (−1, 11).

Solution:

a. For **v** = (−1, 2), you have

$$T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3).$$

Example 1 – Solution

b. If **v** = (0, 0), then

$$T(0, 0) = (0 - 0, 0 + 2(0)) = (0, 0).$$

c. If $T(\mathbf{v}) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$, then

$$v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11.$$

This system of equations has the unique solution $v_1 = 3$ and $v_2 = 4$. So, the preimage of $(-1, 11)$ is the set in R^2 consisting of the single vector (3, 4).

Linear Transformations

Linear Transformations (1 of 4)

Definition of a Linear Transformation

Let V and W be vector spaces. The function

$$T: V \to W.$$

is a **linear transformation** of V into W when the two properties below are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and for any scalar c.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

2. $T(c\mathbf{u}) = cT(\mathbf{u})$

Example 2 – Verifying a Linear Transformation from R^2 into R^2

Show that the function in Example 1 is a linear transformation from R^2 into R^2.

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Solution:

Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ be vectors in R^2 and let c be any real number.

Example 2 – Solution (1 of 2)

Then, using the properties of vector addition and scalar multiplication, you have the two statements below.

1. $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$, so you have

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}). \end{aligned}$$

Example 2 – Solution (2 of 2)

2. $c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$, so you have

$$\begin{aligned} T(c\mathbf{u} &= T(cu_1, cu_2) \\ &= (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}). \end{aligned}$$

So, T is a linear transformation.

Linear Transformations (2 of 4)

Theorem 6.1 Properties of Linear Transformations

Let T be a linear transformation from V into W, where $\mathbf{u}$ and $\mathbf{v}$ are in V. Then the properties listed below are true.

1. $T(\mathbf{0}) = \mathbf{0}$

2. $T(-\mathbf{v}) = -T(\mathbf{v})$

3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$, then

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \dots + c_nT(\mathbf{v}_n). \end{aligned}$$

Example 4 – Linear Transformations and Bases

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$T(1, 0, 0) = (2, -1, 4)$

$T(0, 1, 0) = (1, 5, -2)$

$T(0, 0, 1) = (0, 3, 1)$.

Find $T(2, 3, -2)$.

Example 4 – Solution

$(2, 3, -2) = 2(1, 0, 0) + 3(0, 1, 0) - 2(0, 0, 1)$, so use Property 4 of Theorem 6.1 to write

$$T(2, 3, -2) = 2T(1, 0, 0) + 3T(0, 1, 0) - 2T(0, 0, 1)$$

$$= 2(2, -1, 4) + 3(1, 5, -2) - 2(0, 3, 1)$$

$$= (7, 7, 0).$$

Example 5 – A Linear Transformation Defined by a Matrix

Define the function $T: R^2 \rightarrow R^3$ as

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

a. Find $T(v)$ when $v = (2, -1)$.

b. Show that T is a linear transformation from R^2 into R^3.

Example 5 – Solution (1 of 2)

a. v = (2, −1), so you have

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underset{\substack{\uparrow \\ \text{Vector} \\ \text{in } R^2}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{Vector} \\ \text{in } R^3}}{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$$

which means that $T(2, -1) = (6, 3, 0)$.

Example 5 – Solution (2 of 2)

b. Begin by observing that T maps a vector in R^2 to a vector in R^3. To show that T is a linear transformation. For any vectors **u** and **v in** R^2, the distributive property of matrix multiplication over addition produces

$$T(u + v) = A(u + v) = Au + Av = T(u) + T(v).$$

Similarly, for any vector **u in** R^2 and any scalar c, *the* commutative property of scalar multiplication with matrix multiplication produces

$$T(cu) = A(cu) = c(Au) = cT(u).$$

Linear Transformations (3 of 4)

Theorem 6.2 Linear Transformation Given by a Matrix

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m. In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent the vectors in R^n and $m \times 1$ matrices represent the vectors in R^m.

Example 6 – Linear Transformations Given by Matrices

Consider the linear transformation $T: R^n \rightarrow R^m$ defined by *T*(***v***) = ***Av***. Find the dimensions of R^n and R^m for the linear transformation represented by each matrix.

a. $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$

Example 6 – Solution (1 of 2)

a. The size of this matrix is 3 × 3, so it defines a linear transformation from R^3 into R^3.

$$A\mathbf{v} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\text{Vector in } R^3} = \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\text{Vector in } R^3}$$

Example 6 – Solution (2 of 2)

b. The size of this matrix is 3 × 2, so it defines a linear transformation from R^2 into R^3.

c. The size of this matrix is 2 × 4, so it defines a linear transformation from R^4 into R^2.

Example 10 – The Differential Operator (Calculus)

Let $C'[a, b]$ be the set of all functions whose derivatives are continuous on $[a, b]$. Show that the differential operator D_x defines a linear transformation from $C'[a, b]$ into $C[a, b]$.

Solution:

Using operator notation, you can write

$$D_x(f) = \frac{d}{dx}[f]$$

where f is in $C'[a, b]$. To show that D_x is a linear transformation, you must use calculus.

Example 10 – Solution (1 of 2)

Specifically, the derivative of the sum of two differentiable functions is equal to the sum of their derivatives, so you have

$$D_x(f + g) = \frac{d}{dx}[f + g] = \frac{d}{dx}[f] + \frac{d}{dx}[g] = D_x(f) + D_x(g)$$

where g is also in $C'[a, b]$. Similarly, the derivative of a scalar multiple cf of a differentiable function is equal to the scalar multiple of the derivative, so you have

$$D_x(cf) = \frac{d}{dx}[cf] = c\left(\frac{d}{dx}[f]\right) = cD_x(f).$$

Example 10 – Solution (2 of 2)

The sum of two continuous functions is continuous, and the scalar multiple of a continuous function is continuous, so D_x is a linear transformation from $C'[a, b]$ into $C[a, b]$.

Linear Transformations (4 of 4)

The linear transformation D_x is called the **differential operator.** For polynomials, the differential operator is a linear transformation from P_n into P_{n-1} because the derivative of a polynomial function of degree $n \geq 1$ is a polynomial function of degree $n - 1$. That is,

$$D_x(a_0 + a_1x + \cdot \cdot \cdot + a_nx^n) = a_1 + \cdot \cdot \cdot + na_nx^{n-1}.$$

Example 11 – The Definite Integral as a Linear Transformation (Calculus)

Consider $T: P \rightarrow R$ defined by

$$T(p) = \int_a^b p(x)\,dx$$

where *p* is a polynomial function. Show that *T* is a linear transformation from *P*, the vector space of polynomial functions, into *R*, the vector space of real numbers.

Example 11 – Solution

Using properties of definite integrals, you can write

$$T(p+q) = \int_a^b [p(x) + q(x)]\,dx = \int_a^b p(x)\,dx + \int_a^b q(x)\,dx = T(p) + T(q)$$

where *q* is a polynomial function, and

$$T(cp) = \int_a^b [cp(x)]\,dx = c\int_a^b p(x)\,dx = cT(p)$$

where *c* is a scalar. So, *T* is a linear transformation.