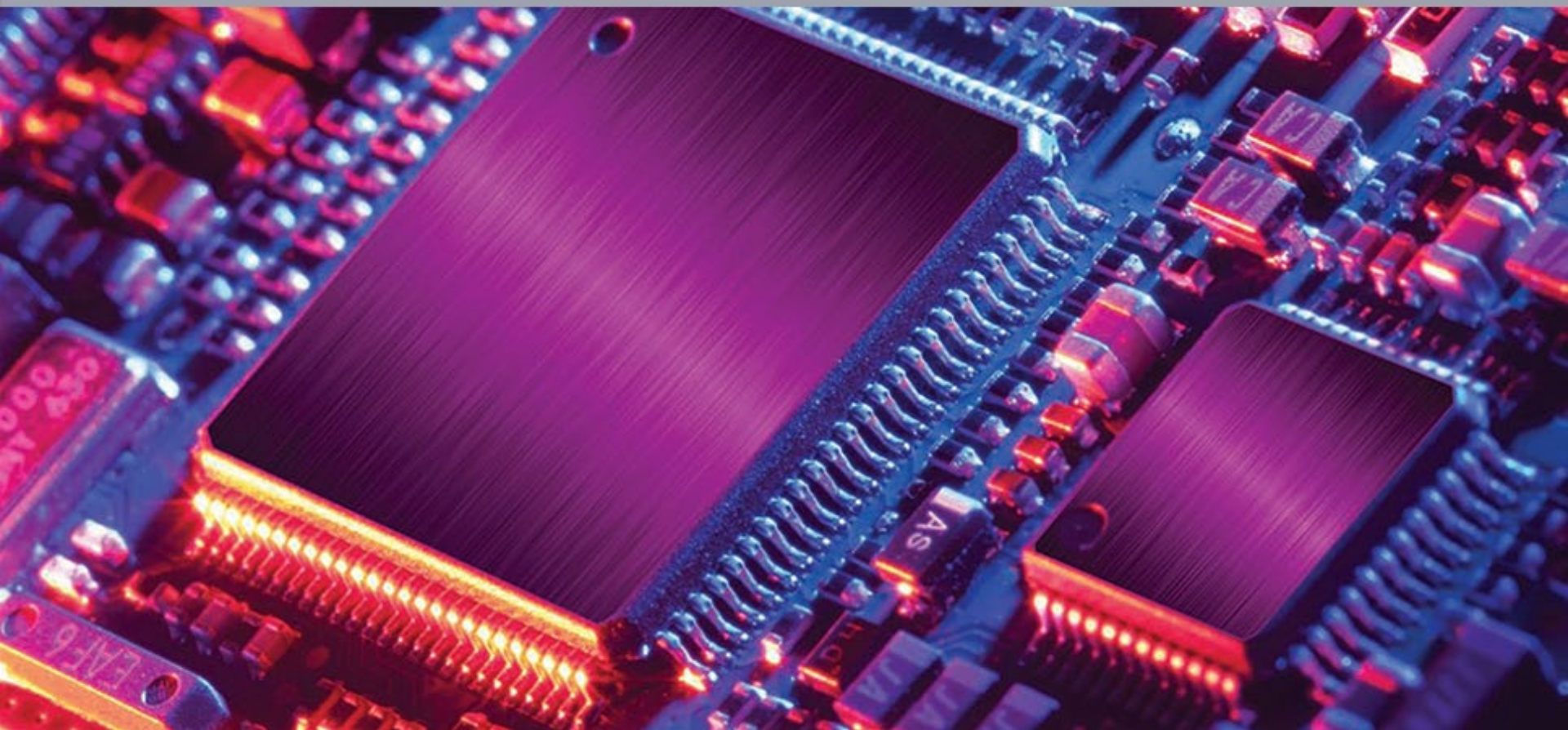


7

The Laplace Transform



7.1

Definition of the Laplace Transform

Definition of the Laplace Transform (5 of 11)

A Definition

The function $K(s, t)$ in (1) is called the **kernel** of the transform. The choice $K(s, t) = e^{-st}$ as the kernel gives us an especially important integral transform.

Definition 7.1.1 Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (2)$$

is said to be the **Laplace transform** of f , provided that the integral converges.

Definition of the Laplace Transform (6 of 11)

$\mathcal{L}$ is a Linear Transform

For a linear combination of functions we can write

$$\int_0^{\infty} e^{-st} [\alpha f(t) + \beta g(t)] dt = \alpha \int_0^{\infty} e^{-st} f(t) dt + \beta \int_0^{\infty} e^{-st} g(t) dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s). \quad (3)$$

Because of the property given in (3), $\mathcal{L}$ is said to be a **linear transform**.

Definition of the Laplace Transform (7 of 11)

Theorem 7.1.1 Transforms of Some Basic Functions

$$(a) \mathcal{L}\{1\} = \frac{1}{s}$$

$$(b) \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \dots$$

$$(c) \mathcal{L}\{e^{at}\} = \frac{1}{s - a}$$

$$(d) \mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

$$(e) \mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$(f) \mathcal{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

$$(g) \mathcal{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$$

Definition of the Laplace Transform (11 of 11)

Theorem 7.1.3 Behavior of $F(s)$ as $s \rightarrow \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s \rightarrow \infty} F(s) = 0$.

7.2

Inverse Transforms and Transforms of Derivatives



7.2.1 Inverse Transforms

7.2.1 Inverse Transforms (1 of 4)

The Inverse Problem

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

7.2.1 Inverse Transforms (2 of 4)

Theorem 7.2.1 Some Inverse Transforms

$$(a) \quad 1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$(b) \quad t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \dots$$

$$(c) \quad e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$(d) \quad \sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2 + k^2}\right\}$$

$$(e) \quad \cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\}$$

$$(f) \quad \sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2 - k^2}\right\}$$

$$(g) \quad \cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2 - k^2}\right\}$$

Example 1 – Applying Theorem 7.2.1

Evaluate (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 7}\right\}$.

Solution:

(a) To match the form given in part (b) of Theorem 7.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by $4!$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!} \mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24} t^4.$$

Example 1 – Solution (1 of 1)

(b) To match the form *given* in part (d) of Theorem 7.2.1, we identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing by $\sqrt{7}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 7}\right\} = \frac{1}{\sqrt{7}} \mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2 + 7}\right\} = \frac{1}{\sqrt{7}} \sin \sqrt{7}t.$$

7.2.1 Inverse Transforms (3 of 4)

$\mathcal{L}^{-1}$ is a Linear Transform

The inverse Laplace transform is also a linear transform; that is, for constants α and β

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}, \quad (1)$$

where F and G are the transforms of some functions f and g .

Example 2 – Termwise Division and Linearity

Evaluate $\mathcal{L}^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\}$.

Solution:

We first rewrite the given function of s as two expressions by means of termwise division and then use (1):

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\} &= \mathcal{L}^{-1}\left\{\frac{-2s}{s^2 + 4} + \frac{6}{s^2 + 4}\right\} = -2 \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + \frac{6}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} \quad (2) \\ &= -2 \cos 2t + 3 \sin 2t. \quad \leftarrow \begin{array}{l} \text{parts (e) and (d)} \\ \text{of Theorem 7.2.1 with } k = 2 \end{array} \end{aligned}$$

7.2.1 Inverse Transforms (4 of 4)

Partial Fractions

Partial fractions play an important role in finding inverse Laplace transforms. The decomposition of a rational expression into component fractions can be done quickly by means of a single command on most computer algebra systems.

The next example illustrates partial fraction decomposition in the case when the denominator of $F(s)$ is factorable into *distinct linear factors*.

Example 3 – Partial Fractions: Distinct Linear Factors

Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)}\right\}.$

Solution:

There exist unique real constants A , B , and C so that

$$\begin{aligned}\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} &= \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s + 4} \\ &= \frac{A(s - 2)(s + 4) + B(s - 1)(s + 4) + C(s - 1)(s - 2)}{(s - 1)(s - 2)(s + 4)}.\end{aligned}$$

Example 3 – Solution (1 of 3)

Since the denominators are identical, the numerators are identical:

$$s^2 + 6s + 9 = A(s - 2)(s + 4) + B(s - 1)(s + 4) + C(s - 1)(s - 2). \quad (3)$$

By comparing coefficients of powers of s on both sides of the equality, we know that (3) is equivalent to a system of three equations in the three unknowns A , B , and C .

Example 3 – Solution (2 of 3)

However, there is a shortcut for determining these unknowns. If we set $s = 1$, $s = 2$, and $s = -4$ in (3), we obtain, respectively,

$$16 = A(-1)(5), \quad 25 = B(1)(6), \quad \text{and} \quad 1 = C(-5)(-6),$$

and so $A = -\frac{16}{5}$, $B = \frac{25}{6}$, and $C = \frac{1}{30}$.

Example 3 – Solution (3 of 3)

Hence the partial fraction decomposition is

$$\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} = -\frac{16/5}{s - 1} + \frac{25/6}{s - 2} + \frac{1/30}{s + 4}, \quad (4)$$

and thus, from the linearity of $\mathcal{L}^{-1}$ and part (c) of Theorem 7.2.1,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)}\right\} &= -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s + 4}\right\} \\ &= -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}. \end{aligned} \quad (5)$$



7.2.2 Transforms of Derivatives

7.2.2 Transforms of Derivatives (1 of 7)

Theorem 7.2.2 Transform of a Derivative

If $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$, and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Example 5 – Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$.

Solution:

We take the sum of the transforms of each term, use (6) and (7), use the given initial conditions, use (c) of Theorem 7.1.1, and then solve for $Y(s)$:

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

Example 5 – Solution (1 of 1)

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}. \quad (14)$$

The details of the partial fraction decomposition of $Y(s)$ in (14) have already been carried out in Example 3. In view of the results in (4) and (5) we have the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

7.3

Operational Properties

I

7.3.1 Translation on the s -Axis

7.3.1 Translation on the s-Axis (1 of 2)

Theorem 7.3.1 First Translation Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

Example 1 – Using the First Translation Theorem

Evaluate (a) $\mathcal{L}\{e^{5t}t^3\}$ (b) $\mathcal{L}\{e^{-2t}\cos 4t\}$.

Solution:

The results follow from Theorems 7.1.1 and 7.3.1.

$$(a) \mathcal{L}\{e^{5t}t^3\} = \mathcal{L}\{t^3\}|_{s \rightarrow s-5} = \frac{3!}{s^4} \Big|_{s \rightarrow s-5} = \frac{6}{(s-5)^4}$$

$$(b) \mathcal{L}\{e^{-2t}\cos 4t\} = \mathcal{L}\{\cos 4t\}|_{s \rightarrow s-(-2)} = \frac{s}{s^2 + 16} \Big|_{s \rightarrow s+2} = \frac{s+2}{(s+2)^2 + 16}$$

7.3.1 Translation on the s-Axis (2 of 2)

Inverse Form of Theorem 7.3.1

To compute the inverse of $F(s - a)$, we must recognize $F(s)$, find $f(t)$ by taking the inverse Laplace transform of $F(s)$, and then multiply $f(t)$ by the exponential function e^{at} . This procedure can be summarized symbolically in the following manner:

$$\mathcal{L}^{-1}\{F(s - a)\} = \mathcal{L}^{-1}\{F(s)|_{s \rightarrow s-a}\} = e^{at}f(t), \quad (1)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Example 2 – Partial Fractions: Repeated Linear Factors

Evaluate

$$(a) \mathcal{L}^{-1}\left\{\frac{2s + 5}{(s - 3)^2}\right\} \qquad (b) \mathcal{L}^{-1}\left\{\frac{s/2 + 5/3}{s^2 + 4s + 6}\right\}.$$

Solution:

(a) A repeated linear factor is a term $(s - a)^n$, where a is a real and n is a positive integer ≥ 2 . We know that if $(s - a)^n$ appears in the denominator of a rational expression, then the assumed decomposition contains n partial fractions with constant numerators and denominators $s - a$, $(s - a)^2, \dots, (s - a)^n$. Hence with $a = 3$ and $n = 2$ we write

$$\frac{2s + 5}{(s - 3)^2} = \frac{A}{s - 3} + \frac{B}{(s - 3)^2}.$$

Example 2 – Solution (1 of 5)

By putting the two terms on the right-hand side over a common denominator, we obtain the numerator
 $2s + 5 = A(s - 3) + B$ and this identity yields $A = 2$ and $B = 11$.

Therefore

$$\frac{2s + 5}{(s - 3)^2} = \frac{2}{s - 3} + \frac{11}{(s - 3)^2} \quad (2)$$

and

$$\mathcal{L}^{-1}\left\{\frac{2s + 5}{(s - 3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s - 3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s - 3)^2}\right\}. \quad (3)$$

Example 2 – Solution (2 of 5)

Now $1/(s - 3)^2$ is $F(s) = 1/s^2$ shifted three units to the right. Since $\mathcal{L}^{-1}\{1/s^2\} = t$, it follows from (1) that.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s - 3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} \mid_{s \rightarrow s-3}\right\} = e^{3t}t.$$

Finally, (3) is

$$\mathcal{L}^{-1}\left\{\frac{2s + 5}{(s - 3)^2}\right\} = 2e^{3t} + 11e^{3t}t. \quad (4)$$

Example 2 – Solution (3 of 5)

(b) To start, observe that the $s^2 + 4s + 6$ has no real zeros and so has no real linear factors. In this situation we *complete the square*:

$$\frac{s/2 + 5/3}{s^2 + 4s + 6} = \frac{s/2 + 5/3}{(s + 2)^2 + 2}. \quad (5)$$

Our goal here is to recognize the expression on the right-hand side as some Laplace transform $F(s)$ in which s has been replaced throughout by $s + 2$. What we are trying to do is analogous to working part (b) of Example 1 backwards. The denominator in (5) is already in the correct form—that is, $s^2 + 2$ with s replaced by $s + 2$.

Example 2 – Solution (4 of 5)

However, we must fix up the numerator by manipulating the constants: $\frac{1}{2}s + \frac{5}{3} = \frac{1}{2}(s + 2) + \frac{5}{3} - \frac{2}{2} = \frac{1}{2}(s + 2) + \frac{2}{3}$.

Now by term wise division, the linearity of $\mathcal{L}^{-1}$, parts (d) and (e) of Theorem 7.2.1, and finally (1),

$$\frac{s/2 + 5/3}{(s + 2)^2 + 2} = \frac{\frac{1}{2}(s + 2) + \frac{2}{3}}{(s + 2)^2 + 2} = \frac{1}{2} \frac{s + 2}{(s + 2)^2 + 2} + \frac{2}{3} \frac{1}{(s + 2)^2 + 2}$$

$$\mathcal{L}^{-1}\left\{\frac{s/2 + 5/3}{s^2 + 4s + 6}\right\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{s + 2}{(s + 2)^2 + 2}\right\} + \frac{2}{3} \mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2 + 2}\right\}$$

Example 2 – Solution (5 of 5)

$$= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2} \middle|_{s \rightarrow s+2} \right\} + \frac{2}{3\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{2}}{s^2 + 2} \middle|_{s \rightarrow s+2} \right\} \quad (6)$$

$$= \frac{1}{2} e^{-2t} \cos \sqrt{2}t + \frac{\sqrt{2}}{3} e^{-2t} \sin \sqrt{2}t. \quad (7)$$



7.3.2 Translation on the t -Axis

7.3.2 Translation on the t -Axis (1 of 14)

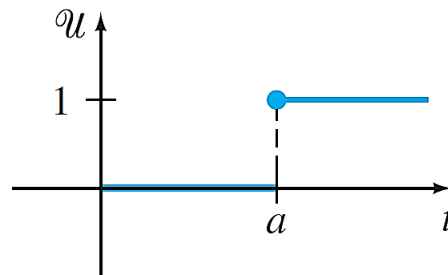
Definition 7.3.1 Unit Step Function

The **unit step function** $\mathcal{U}(t - a)$ is defined to be

$$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a. \end{cases}$$

7.3.2 Translation on the t -Axis (2 of 14)

Notice that we define $\mathcal{U}(t - a)$ only on the nonnegative t -axis, since this is all that we are concerned with in the study of the Laplace transform. In a broader sense $\mathcal{U}(t - a) = 0$ for $t < a$. The graph of $\mathcal{U}(t - a)$ is given in figure. In the case when $a = 0$, we take $\mathcal{U}(t) = 1$ for $t \geq 0$.



Graph of unit step function

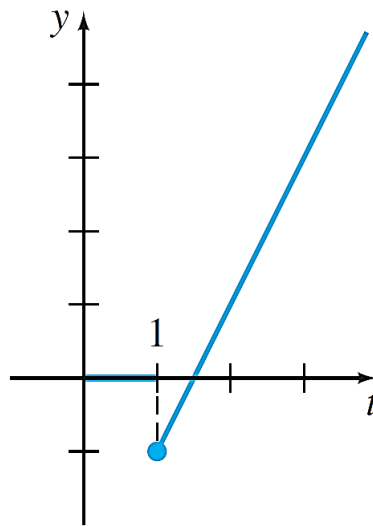
Figure 7.3.2

7.3.2 Translation on the t -Axis (3 of 14)

When a function f defined for $t \geq 0$ is multiplied by $\mathcal{U}(t - a)$ the unit step function “turns off” a portion of the graph of that function. For example, consider the function $f(t) = 2t - 3$. To “turn off” the portion of the graph of f for $0 \leq t \leq 1$, we simply form the product $(2t - 3)\mathcal{U}(t - 1)$.

7.3.2 Translation on the t -Axis (4 of 14)

See figure. In general the graph of $f(t) \mathcal{U}(t - a)$ is 0 (off) for $0 \leq t < a$ and is the portion of the graph of f (on) for $t \geq a$.

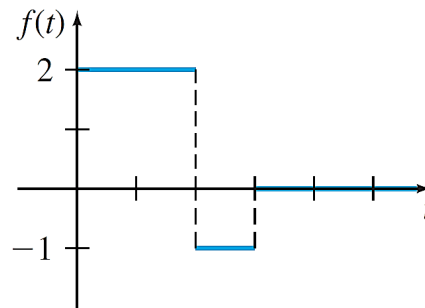


Function is $f(t) = (2t - 3)\mathcal{U}(t - 1)$.

Figure 7.3.3

7.3.2 Translation on the t -Axis (5 of 14)

The unit step function can also be used to write piecewise-defined functions in a compact form. For example, if we consider $0 \leq t < 2$, $2 \leq t < 3$, and $t \geq 3$ and the corresponding values of $\mathcal{U}(t - 2)$ and $\mathcal{U}(t - 3)$, it should be apparent that the piecewise defined function shown in figure is the same as $f(t) = 2 - 3\mathcal{U}(t - 2) + \mathcal{U}(t - 3)$.



Function is $f(t) = 2 - 3\mathcal{U}(t - 2) + \mathcal{U}(t - 3)$

Figure 7.3.4

7.3.2 Translation on the t -Axis (6 of 14)

Also, a general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} \quad (9)$$

is the same as

$$f(t) = g(t) - g(t)\mathcal{U}(t - a) + h(t)\mathcal{U}(t - a). \quad (10)$$

7.3.2 Translation on the t -Axis (7 of 14)

Similarly, a function of the type

$$f(t) = \begin{cases} 0, & 0 \leq t < a \\ g(t), & a \leq t < b \\ 0, & t \geq b \end{cases} \quad (11)$$

can be written

$$f(t) = g(t)[\mathcal{U}(t - a) - \mathcal{U}(t - b)]. \quad (12)$$

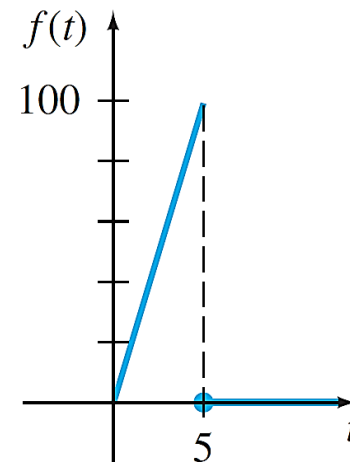
Example 5 – A Piecewise-Defined Function

Express $f(t) = \begin{cases} 20t, & 0 \leq t < 5 \\ 0, & t \geq 5 \end{cases}$ in terms of unit step functions.
Graph.

Solution:

The graph of f is given in figure.
Now from (9) and (10) with $a = 5$,
 $g(t) = 20t$, and $h(t) = 0$ we get

$$f(t) = 20t - 20t \mathcal{U}(t - 5).$$



Function f in Example 5

Figure 7.3.5

7.3.2 Translation on the t -Axis (8 of 14)

Theorem 7.3.2 Second Translation Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

We often wish to find the Laplace transform of just a unit step function. This can be from either Definition 7.1.1 or Theorem 7.3.2. If we identify $f(t) = 1$ in Theorem 7.3.2, then $f(t - a) = 1$, $F(s) = \mathcal{L}\{1\} = 1/s$, and so

$$\mathcal{L}\{\mathcal{U}(t - a)\} = \frac{e^{-as}}{s}. \quad (14)$$

Example 6 – Figure 7.3.4 Revisited

Find the Laplace transform of the function

$$f(t) = 2 - 3\mathcal{U}(t - 2) + \mathcal{U}(t - 3)$$

Solution:

We use f expressed in terms of the unit step function

$$f(t) = 2 - 3\mathcal{U}(t - 2) + \mathcal{U}(t - 3)$$

and the result given in (14):

$$\begin{aligned}\mathcal{L}\{f(t)\} &= 2\mathcal{L}\{1\} - 3\mathcal{L}\{\mathcal{U}(t - 2)\} + \mathcal{L}\{\mathcal{U}(t - 3)\} \\ &= \frac{2}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}.\end{aligned}$$

7.3.2 Translation on the t -Axis (9 of 14)

Inverse Form of Theorem 7.3.2

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, the inverse form of Theorem 7.3.2, $a > 0$, is

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\mathcal{U}(t - a). \quad (15)$$

Example 7 – Using Formula (15)

Evaluate (a) $\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\}.$

Solution:

(a) With the three identifications $a = 2$, $F(s) = 1/(s - 4)$, and $\mathcal{L}^{-1}\{F(s)\} = e^{4t}$, we have from (15)

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)} \mathcal{U}(t-2).$$

7.4

Operational Properties II



7.4.1 Derivatives of a Transform

7.4.1 Derivatives of a Transform (1 of 2)

Theorem 7.4.1 Derivatives of Transforms

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \dots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Example 1 – Using Theorem 7.4.1

Evaluate $\mathcal{L}\{t \sin kt\}$.

Solution:

With $f(t) = \sin kt$, $F(s) = k/(s^2 + k^2)$, and $n = 1$,

Theorem 7.4.1 gives

$$\mathcal{L}\{t \sin kt\} = -\frac{d}{ds} \mathcal{L}\{\sin kt\} = -\frac{d}{ds} \left(\frac{k}{s^2 + k^2} \right) = \frac{2ks}{(s^2 + k^2)^2}.$$

7.4.1 Derivatives of a Transform (2 of 2)

Note

To find transforms of functions $t^n e^{at}$ we can use either Theorem 7.3.1 or Theorem 7.4.1. For example,

$$\text{Theorem 7.3.1: } \mathcal{L}\{te^{3t}\} = \mathcal{L}\{t\}_{s \rightarrow s-3} = \frac{1}{s^2} \Big|_{s \rightarrow s-3} = \frac{1}{(s-3)^2}.$$

$$\text{Theorem 7.4.1: } \mathcal{L}\{te^{3t}\} = -\frac{d}{ds} \mathcal{L}\{e^{3t}\} = -\frac{d}{ds} \frac{1}{s-3} = (s-3)^{-2} = \frac{1}{(s-3)^2}.$$

Example 2 – An Initial-Value Problem

Solve $x'' + 16x = \cos 4t$, $x(0) = 0$, $x'(0) = 1$.

Solution:

The initial-value problem could describe the forced, undamped, and resonant motion of a mass on a spring. The mass starts with an initial velocity of 1 ft/s in the downward direction from the equilibrium position. Transforming the differential equation gives

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16} \quad \text{or} \quad X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}.$$

Example 2 – Solution (1 of 1)

Now we just saw in Example 1 that

$$\mathcal{L}^{-1}\left\{\frac{2ks}{(s^2 + k^2)^2}\right\} = t \sin kt, \quad (1)$$

and so with the identification $k = 4$ in (1) and in part (d) of Theorem 7.2.1, we obtain

$$\begin{aligned} x(t) &= \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8} \mathcal{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\} \\ &= \frac{1}{4} \sin 4t + \frac{1}{8} t \sin 4t. \end{aligned}$$



7.4.2 Transforms of Integrals

7.4.2 Transforms of Integrals (2 of 11)

Convolution Theorem

Theorem 7.4.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = F(s)G(s).$$

7.4.2 Transforms of Integrals (4 of 11)

Transform of an Integral

When $g(t) = 1$ and $\mathcal{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}. \quad (7)$$

The inverse form of (7),

$$\int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}, \quad (8)$$

can be used in lieu of partial fractions when s^n is a factor denominator and $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is easy to integrate.

7.5

The Dirac Delta Function

The Dirac Delta Function (1 of 6)

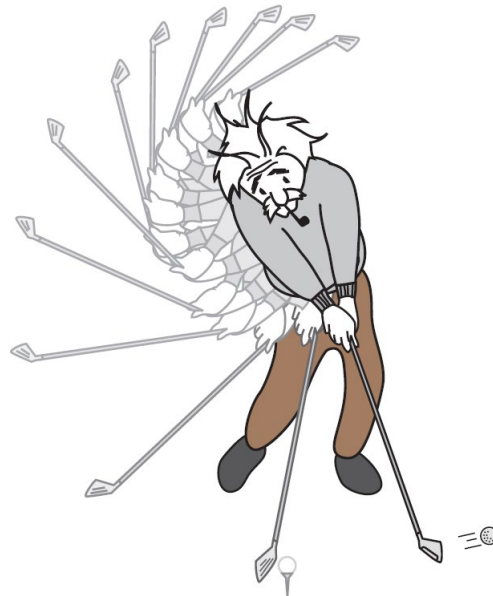
Unit Impulse

Mechanical systems are often acted on by an external force (or electromotive force in an electrical circuit) of large magnitude that acts only for a very short period of time.

For example, a vibrating airplane wing could be struck by lightning, a mass on a spring could be given a sharp blow by a ball peen hammer, and a ball (baseball, golf ball, tennis ball) could be sent soaring when struck violently by some kind of club (baseball bat, golf club, tennis racket).

The Dirac Delta Function (2 of 6)

See figure. The graph of the piecewise-defined function



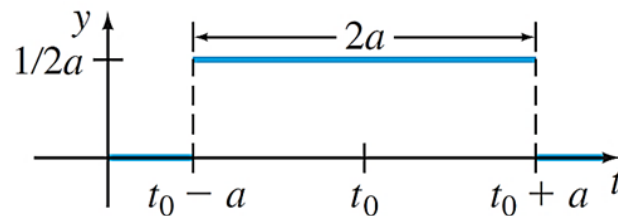
A golf club applies a force of large magnitude on the ball for a very short period of time

Figure 7.5.1

The Dirac Delta Function (3 of 6)

$$\delta_a(t - t_0) = \begin{cases} 0, & 0 \leq t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \leq t < t_0 + a \\ 0, & t \geq t_0 + a, \end{cases} \quad (1)$$

$a > 0$, $t_0 > 0$, shown in figure,



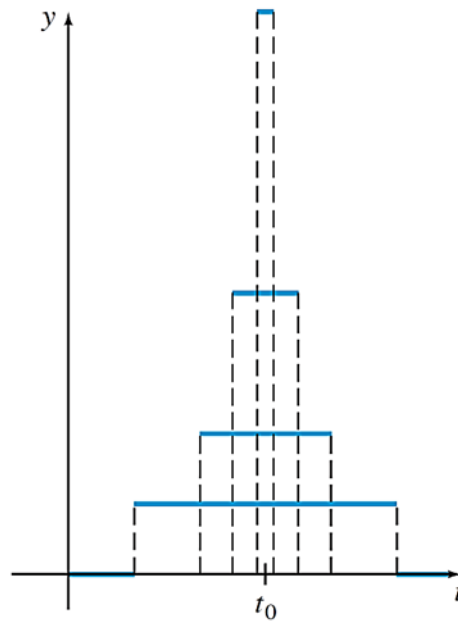
(a) graph of $\delta_a(t - t_0)$

Figure 7.5.2(a)

could serve as a model for such a force. For a small value of a , $\delta_a(t - t_0)$ is essentially a constant function of large magnitude that is “on” for just a very short period of time, around t_0 .

The Dirac Delta Function (4 of 6)

The behavior of $\delta_a(t - t_0)$ as $a \rightarrow 0$ is illustrated in figure.



(b) behavior of δ_a as $a \rightarrow 0$

Unit impulse

Figure 7.5.2(b)

The Dirac Delta Function (5 of 6)

The function $\delta_a(t - t_0)$ is called a **unit impulse**, because it possesses the integration property $\int_0^\infty \delta_a(t - t_0) dt = 1$.

Dirac Delta Function

In practice it is convenient to work with another type of unit impulse, a “function” that approximates $\delta_a(t - t_0)$ and is defined by the limit

$$\delta(t - t_0) = \lim_{a \rightarrow 0} \delta_a(t - t_0). \quad (2)$$

The latter expression, which is not a function at all, can be characterized by the two properties

$$(i) \delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases} \quad \text{and} \quad (ii) \int_0^\infty \delta(t - t_0) dt = 1.$$

The Dirac Delta Function (6 of 6)

The unit impulse $\delta_a(t - t_0)$ is called the **Dirac delta function**.

It is possible to obtain the Laplace transform of the Dirac delta function by the formal assumption that $\mathcal{L}\{\delta(t - t_0)\} = \lim_{a \rightarrow 0} \mathcal{L}\{\delta_a(t - t_0)\}$.

Theorem 7.5.1 Transform of the Dirac Delta Function

For $t_0 > 0$, $\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$. (3)

Example 1 – Two Initial-Value Problems

Solve $y'' + y = 4 \delta(t - 2\pi)$ subject to

(a) $y(0) = 1, \quad y'(0) = 0$ (b) $y(0) = 0, \quad y'(0) = 0$.

The two initial-value problems could serve as models for describing the motion of a mass on a spring moving in a medium in which damping is negligible. At $t = 2\pi$ the mass is given a sharp blow. In (a) the mass is released from rest 1 unit below the equilibrium position. In (b) the mass is at rest in the equilibrium position.

Example 1 – Solution (1 of 4)

(a) From (3) the Laplace transform of the differential equation is

$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s} \quad \text{or} \quad Y(s) = \frac{s}{s^2 + 1} + \frac{4e^{-2\pi s}}{s^2 + 1}.$$

Using the inverse form of the second translation theorem, (15), we find

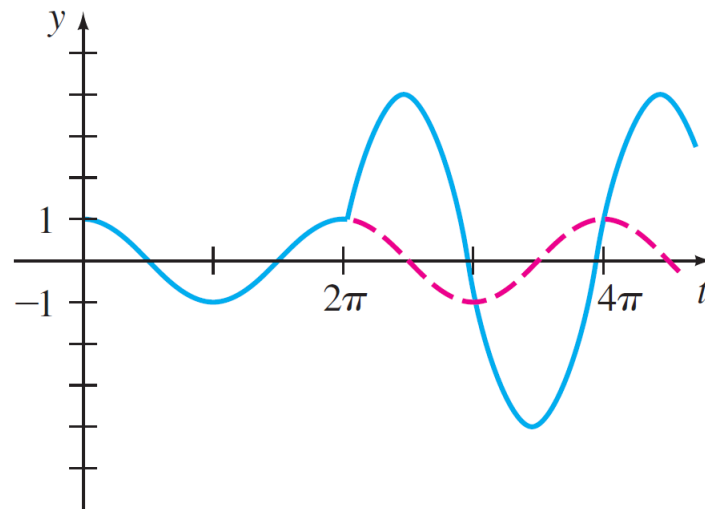
$$y(t) = \cos t + 4 \sin(t - 2\pi) \mathcal{U}(t - 2\pi).$$

Since $\sin(t - 2\pi) = \sin t$, the foregoing solution can be written as

$$y(t) = \begin{cases} \cos t, & 0 \leq t < 2\pi \\ \cos t + 4 \sin t, & t \geq 2\pi. \end{cases} \quad (5)$$

Example 1 – Solution (2 of 4)

In figure we see from the graph of (5) that the mass is exhibiting simple harmonic motion until it is struck at $t = 2\pi$. The influence of the unit impulse is to increase the amplitude of vibration to $\sqrt{17}$ for $t > 2\pi$.



Mass is struck at $t = 2\pi$ in part (a) of Example 1

Figure 7.5.3

Example 1 – Solution (3 of 4)

(b) In this case the transform of the equation is simply

$$Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1},$$

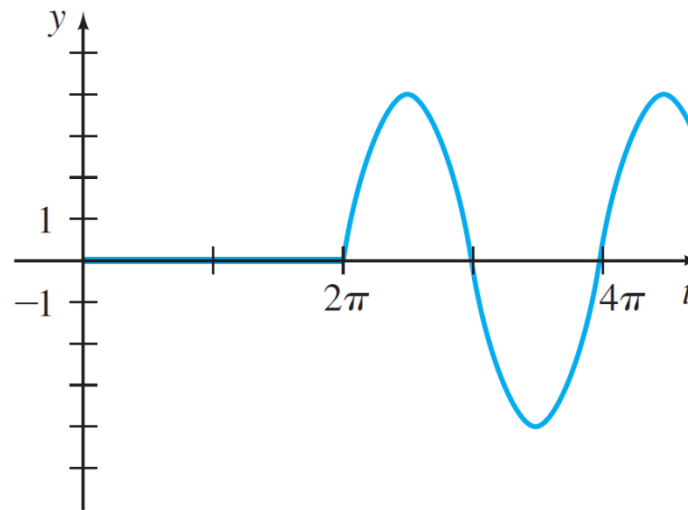
and so

$$y(t) = 4 \sin(t - 2\pi) \mathcal{U}(t - 2\pi)$$

$$= \begin{cases} 0, & 0 \leq t < 2\pi \\ 4 \sin t, & t \geq 2\pi. \end{cases} \quad (6)$$

Example 1 – Solution (4 of 4)

The graph of (6) in figure shows, as we would expect from the initial conditions that the mass exhibits no motion until it is struck at $t = 2\pi$.



No motion until mass is struck at $t = 2\pi$ in part (b) of Example 1

Figure 7.5.4

7.6

Systems of Linear Differential Equations

Systems of Linear Differential Equations (1 of 8)

Introduction

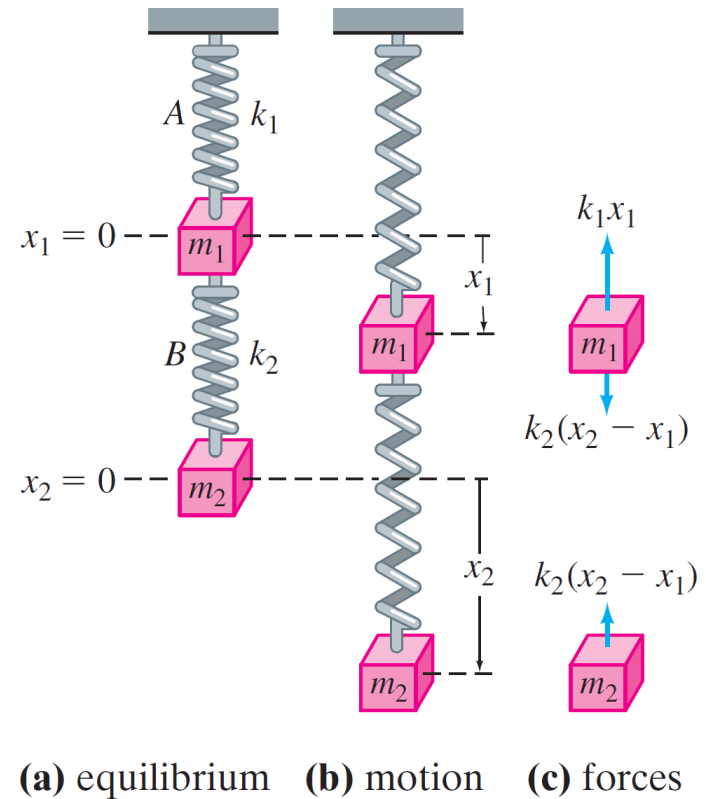
When initial conditions are specified, the Laplace transform of each equation in a system of linear differential equations with constant coefficients reduces the system of DEs to a set of simultaneous algebraic equations in the transformed functions.

We solve the system of algebraic equations for each of the transformed functions and then find the inverse Laplace transforms in the usual manner.

Systems of Linear Differential Equations (2 of 8)

Coupled Springs

Two masses m_1 and m_2 are connected to two springs A and B of negligible mass having spring constants k_1 and k_2 , respectively. In turn the two springs are attached as shown in figure.



Coupled spring/mass system

Figure 7.6.1

Systems of Linear Differential Equations (3 of 8)

Let $x_1(t)$ and $x_2(t)$ denote the vertical displacements of the masses from their equilibrium positions. When the system is in motion, spring B is subject to both an elongation and a compression; hence its net elongation is $x_2 - x_1$.

Therefore it follows from Hooke's law that springs A and B exert forces $-k_1x_1$ and $k_2(x_2 - x_1)$, respectively, on m_1 . If no external force is impressed on the system and if no damping force is present, then the net force on m_1 is $-k_1x_1 + k_2(x_2 - x_1)$. By Newton's second law we can write

$$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1).$$

Systems of Linear Differential Equations (4 of 8)

Similarly, the net force exerted on mass m_2 is due solely to the net elongation of B ; that is, $-k_2(x_2 - x_1)$. Hence we have

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1).$$

In other words, the motion of the coupled system is represented by the system of simultaneous second-order differential equations

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' &= -k_2(x_2 - x_1). \end{aligned} \tag{1}$$

Example 1 – Coupled Springs

Solve

$$\begin{aligned}x_1'' + 10x_1 - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0\end{aligned}\tag{2}$$

subject to $x_1(0) = 0, x_1'(0) = 1, x_2(0) = 0, x_2'(0) = -1$.

Example 1 – Solution (1 of 5)

The Laplace transform of each equation is

$$\begin{aligned} s^2 X_1(s) - s x_1(0) - x_1'(0) + 10 X_1(s) - 4 X_2(s) &= 0 \\ -4 X_1(s) + s^2 X_2(s) - s x_2(0) - x_2'(0) + 4 X_2(s) &= 0, \end{aligned}$$

where $X_1(s) = \mathcal{L}\{x_1(t)\}$ and $X_2(s) = \mathcal{L}\{x_2(t)\}$. The preceding system the same as

$$\begin{aligned} (s^2 + 10) X_1(s) - 4 X_2(s) &= 1 \\ -4 X_1(s) + (s^2 + 4) X_2(s) &= -1. \end{aligned} \tag{3}$$

Example 1 – Solution (2 of 5)

Solving (3) for $X_1(s)$ and using partial fractions on the result yields

$$X_1(s) = \frac{s^2}{(s^2 + 2)(s^2 + 12)} = -\frac{1/5}{s^2 + 2} + \frac{6/5}{s^2 + 12},$$

and therefore

$$\begin{aligned} x_1(t) &= -\frac{1}{5\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + 2}\right\} + \frac{6}{5\sqrt{12}} \mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2 + 12}\right\} \\ &= -\frac{\sqrt{2}}{10} \sin \sqrt{2}t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}t. \end{aligned}$$

Example 1 – Solution (3 of 5)

Substituting the expression for $X_1(s)$ into the first equation of (3) gives

$$X_2(s) = -\frac{s^2 + 6}{(s^2 + 2)(s^2 + 12)} = -\frac{2/5}{s^2 + 2} - \frac{3/5}{s^2 + 12}$$

and

$$\begin{aligned} x_2(t) &= -\frac{2}{5\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + 2}\right\} - \frac{3}{5\sqrt{12}} \mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2 + 12}\right\} \\ &= -\frac{\sqrt{2}}{5} \sin \sqrt{2}t - \frac{\sqrt{3}}{10} \sin 2\sqrt{3}t. \end{aligned}$$

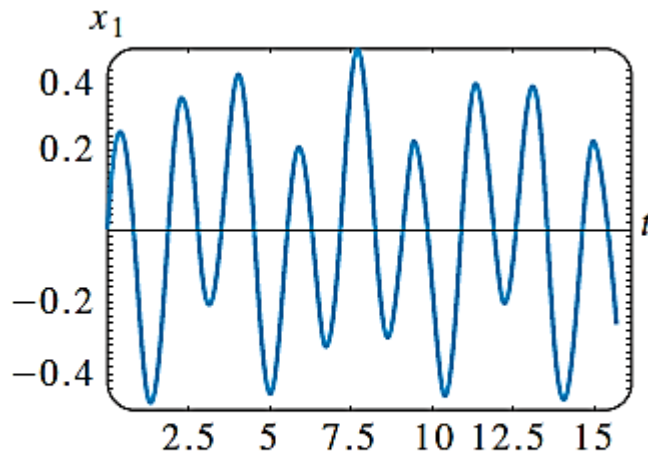
Example 1 – Solution (4 of 5)

Finally, the solution to the given system (2) is

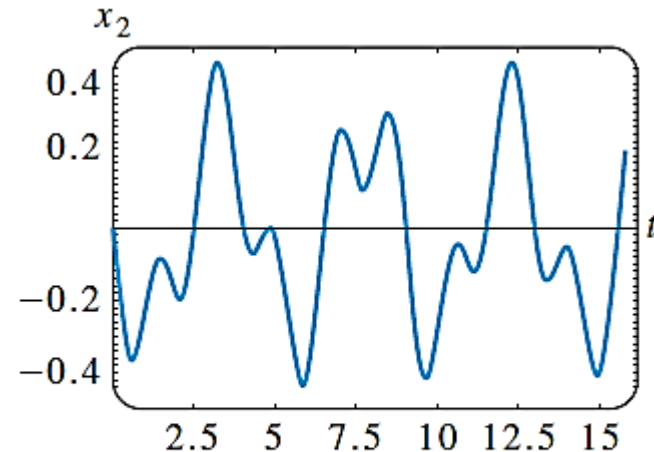
$$\begin{aligned}x_1(t) &= -\frac{\sqrt{2}}{10} \sin \sqrt{2}t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}t \\x_2(t) &= -\frac{\sqrt{2}}{5} \sin \sqrt{2}t - \frac{\sqrt{3}}{10} \sin 2\sqrt{3}t.\end{aligned}\tag{4}$$

Example 1 – Solution (5 of 5)

The graphs of x_1 and x_2 in figure reveal the complicated oscillatory motion of each mass.



(a) plot of $x_1(t)$



(b) plot of $x_2(t)$

Displacements of the two masses in Example 1

Figure 7.6.2